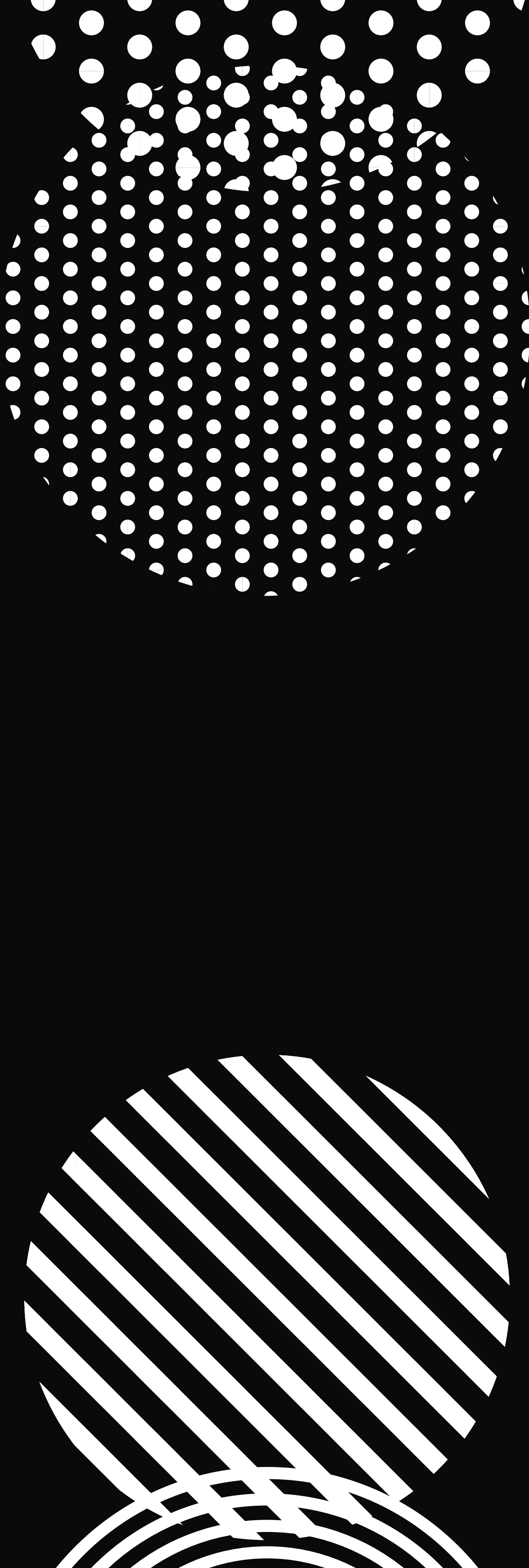


§ $\mathbb{Q}$ Last time

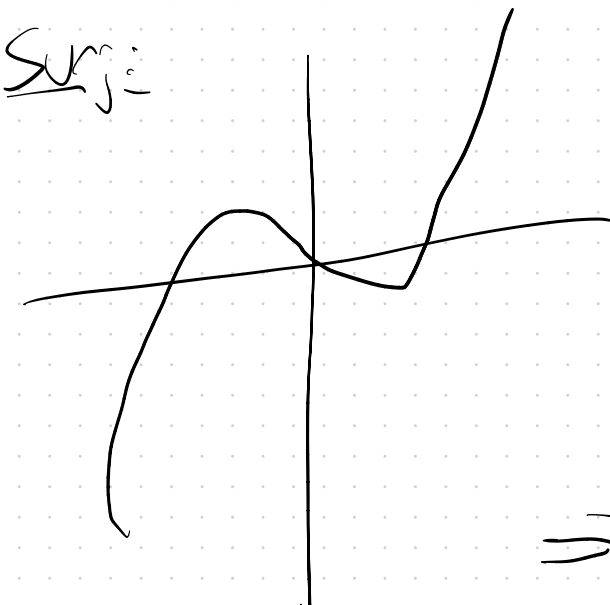
Q: (a) Find an example of a surj $f : \mathbb{R} \to \mathbb{R}$ which is not an inj. and s.t. $f(\mathbb{Q}) \subseteq \mathbb{Q}$ but $f : \mathbb{Q} \to \mathbb{Q}$ is not a surj.

(b) Is this possible if roles of "inj." and "surj." reversed?

A: (a) Take $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x^3-x$.

Not inj: $f(0)=f(1)=f(-1)=0$

Surj:



$$\lim_{x\to\infty} f(x)=\infty$$

$$\lim_{x\to-\infty} f(x)=-\infty$$

$\Rightarrow f$ is surj. by IVT

<u>Restriction to $\mathbb{Q}$:</u> $f(\mathbb{Q}) \subseteq \mathbb{Q}$ as $x^3 - x \in \mathbb{Q}$ if $x \in \mathbb{Q}$.

But,

$$f^{-1}(1) = \{x \in \mathbb{Q} : x^3 - x = 1\} = \emptyset$$

Not entirely obvious, but true

(ii) <u>Not possible</u>: if $f : \mathbb{R} \to \mathbb{R}$ is an inj. then so is $f : \mathbb{Q} \to \mathbb{Q}$!

Thm: Let $f: X \to Y$ be a function. TFAE:

(1) f is a bij.

(2) $\exists$ an inverse function $g: Y \to X$ of f

Moreover, if an inverse function exists it's unique.

Pf: (1) $\Rightarrow$ (2) Define $g: Y \to X$ by

$$g(y) = x \quad \text{if} \quad f(x) = y.$$

Note that as f is bij then is a unique

Such x and So g is well-defined. Note,

$\forall y \in Y$, $f(g(y)) = f(x) = y$. Note $\forall x \in X$

$$g(f(x)) = t \quad \text{s.t.} \quad f(t) = f(x)$$

But, as f is inj. $\Rightarrow$ $t = x$.

(2) $\Rightarrow$ (1) Let g be an inverse of f.

If $f(x_1) = f(x_2) \Rightarrow \underbrace{g(f(x_1))}_{x_1} = \underbrace{g(f(x_2))}_{x_2}$.

So f is inj. For $y \in Y$ $f(g(y)) = y$ so $f^{-1}(y)$ is non-empty. So, f is surj.

Finally we show the inverse g is unique.
Let g_1 and g_2 be two inverses to f. Then
$\forall y \in Y \quad f(g_1(y)) = y = f(g_2(y))$. As f is inj.
$g_1(y) = g_2(y)$. As y was arbitrary, $g_1 = g_2$ $\square$

<u>Def'n</u>: If $f: X \to Y$ is bij., its inverse is denoted $f^{-1}: Y \to Y$

<u>Obs</u>: If f bij. then $f^{-1}(y) = \{f^{-1}(y)\}$ so no confusion.

§1 Cardinality

Definition: For sets X and Y we say X and Y have the same cardinality, written $\#X = \#Y$, if $\exists$ a bij. $f: X \to Y$.

Remark: If $\#X = \#Y$, $\exists$ a bij $f: X \to Y$ but there will essentially always be a non-bij. $g: X \to Y$, e.g., take $g(x) = y_0$ to be constant. $\#X = \#Y$ just means there is at least one bij!

Prop: $\#X = \#Y$ is an equivalence relation.

Pf: Reflexive: $\#X = \#X$ as the identity map $id: X \to X$ is a bij.

Symmetric: If $\#X = \#Y$ then $\exists$ a bij. $f: X \to Y$. Then $f^{-1}: Y \to X$ is a bij. So $\#Y = \#X$.

Transitive: If $\#X = \#Y$ and $\#Y = \#Z$ then $\exists$ bijs. $f: X \to Y$ and $g: Y \to Z$. Then, $g \circ f: X \to Z$ is a bij so $\#X = \#Z$.

Defn: A Cardinal number is an equivalence class of sets under the relation of having the same cardinality.

e.g.: We use the symbol $\aleph_0$ for $\#\mathbb{N}$.

§2 Examples

e.g. $\#(\mathbb{N}-\{0\}) = \aleph_0$ "$\infty = \infty - 1$"

Pf: The map $\mathbb{N} \to \mathbb{N}-\{0\}$ given by $n \mapsto n+1$ is a bij. w/ inverse $m \mapsto m-1$. ∎

e.g. $\#\{\text{Even natural numbers}\} = \aleph_0$ "$\infty = \frac{1}{2}\infty$"

Pf: The map $\mathbb{N} \to \{\text{Even natural numbers}\}$

given by $n \mapsto 2n$ is a bij. w/ inverse $m \mapsto \frac{1}{2} m$. $\boxed{//}$

e.g.) $\# \mathbb{Z} = \aleph_0$

Pf: Define $s : \mathbb{N} \to \mathbb{Z}$, $s(n) = (-1)^{n+1} \left\lfloor \frac{n+1}{2} \right\rfloor$

$0 \mapsto 0$

$1 \mapsto 1$

$2 \mapsto -1$

$3 \mapsto 2$

$4 \mapsto -2$ ...

$5 \mapsto 3$

$6 \mapsto -3$

This is evidently a bij. $\blacksquare$

e.g. $\#\mathbb{Q} = \aleph_0$

Pf: First we define a bij. $\mathbb{N}^+ \to \mathbb{Q}^+$.

But, recall by the Fundamental Theorem of Arithmetic

$$\mathbb{N}^+ = \left\{ \prod_{p \in \mathbb{P}} p^{e_p} : e_p \in \mathbb{N} \right\} \quad \mathbb{Q}^+ = \left\{ \prod_{p \in \mathbb{P}} p^{g_p} : g_p \in \mathbb{Z} \right\}$$

So, define

$$f: \mathbb{N}^+ \to \mathbb{Q}^+, \quad \prod_{p \in \mathbb{P}} p^{c_p} \longmapsto \prod_{p \in \mathbb{P}} p^{s(c_p)}$$

This is a bij w/ inverse

$$\mathbb{Q}^+ \to \mathbb{N}^+, \quad \prod_{p \in \mathbb{P}} p^{g_p} \longmapsto \prod_{p \in \mathbb{P}} p^{s^{-1}(g_p)}$$

We then build a bij $\mathbb{N} \to \mathbb{Q}$ as follows

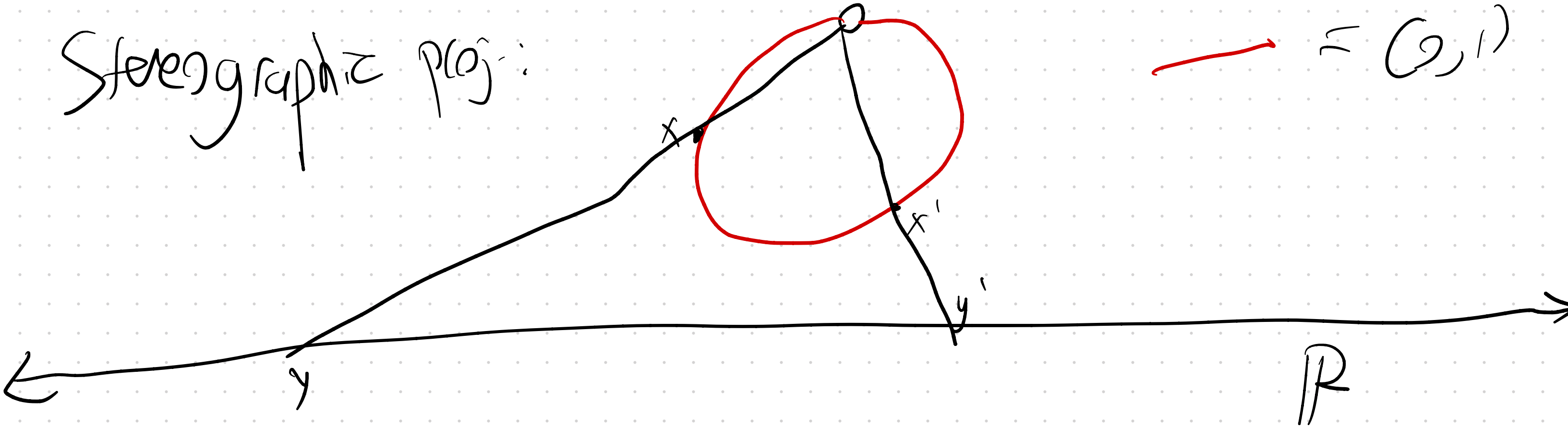
$$
\begin{tikzcd}
\mathbb{N} = & -\mathbb{N}^+ \arrow[d, "-f"'] & \sqcup & \{0\} \arrow[d, "\mathrm{id}"] & \sqcup & \mathbb{N}^+ \arrow[d, "f"] \\
\mathbb{Q} = & -\mathbb{Q}^+ & \sqcup & \{0\} & \sqcup & \mathbb{Q}^+
\end{tikzcd}
$$

$\blacksquare$

e.g. 1 $\quad \# \mathbb{R} = \# (0, 1)$

"Pf": Proof by pictures

Stereographic proj:



§3 Inequality

<u>Defn:</u> Write $\#X \leq \#Y$ if $\exists$ an inj: $X \to Y$ and $\#X < \#Y$ if $\#X \leq \#Y$

and $\# X \neq \# Y$.

Warning: If S is a proper subset of T it is true $\# S \leq \# T$ but not neccesarily true $\# S < \# T$

e.g. $\#(\mathbb{N} - \{0\}) = \# \mathbb{N}$

This sort of characterizes finite sets.

<u>Axiom</u> (Pigeonhole principle): The map

$$\begin{aligned} \mathbb{N} &\longrightarrow \left\{ \begin{matrix} \text{Cardinal} \\ \text{numbers} \end{matrix} \right\} \\ n &\longmapsto \# \{1, \dots, n\} \end{aligned}$$

is an order preserving inj, e.g., if $n > m$ there is no inj. $\{1, \dots, n\} \to \{1, \dots, m\}$.

§4 Countability

Def'n: A set X is

- **finite** if $\# X = n$ for some $n \in \mathbb{N}$,
- **countably infinite** if $\# X = \aleph_0$
- **countable** if $\# X \leq \aleph_0$
- **uncountable** if $\# X > \aleph_0$

Next time:

- $\mathbb{R}$ is uncountable,
- all sets are finite, countably infinite, or uncountable.